Dismantlings and Iterated Clique Graphs

Martín Eduardo Frías Armenta *   Victor Neumann Lara †

December 5^{th}, 2001

Abstract

The clique graph $k(G)$ of a graph G is the intersection graph of the family of cliques of G. A vertex z is dominated by a neighbor w if every neighbor of z is also neighbor of w. We prove in this paper that deletion of dominated points of G does not change the k-behavior of G.

Keywords: Dominated, invariance, $k-$behavior.

2000 Mathematics Subject Clasification: 05C99

1 Introduction.

A clique of a graph G is a maximal complete subgraph of G. The clique graph $k(G)$ of a graph G is the intersection graph of the family of cliquues of G. We say that a graph F is $k-$**periodic** with period p if is it isomorphic to $k^p(F)$ but not to $k^q(F)$ for $1 \leq q < p$. It is said that a graph G is $k-$**stationary** if there is an integer n such that $k^n(G)$ is $k-periodic$. A special case of $k-stationary$ graphs is a $k-$ **null** graph, G is $k-null$ if there is n such that $k^n(G) \cong K_1$. We say that G $k-$**diverges** if $|k^n(G)| \to \infty$ when $n \to \infty$, and in this case we say that $ci(G) = \infty$. The $k-$ **behavior** of G is determinated by the above clasification i.e., it consists in saying if G is $k-stationary$, $k-null$ or k-*divergent*.

Let z, w be elements of graph G. We say that z is dominated by his neighbor w if every neighbor of z is also neighbor of w, that is, $N_G[w] \supseteq N_G[z]$, and we write $w \gtrsim_G z$ (we will omit G when no confusion arisen). We say to that z is dominated if z is dominated by some of its neighbors. Domination induces a preorder. When a graph G do not have dominated points we say that G is irreducible.

The present paper was inspired in Prisner [8] . In that paper Prisner proved that if we eliminate dominated points of graph G until to obtain a irreducible

*Universidad de Sonora, Convenio de Retención Conacyt Ref: 489100-1 Exp: 010260
†Instituto de Matemáticas, UNAM

graph H, and H have no triangles, then there are n such that $k^n(G) \in \{h, k(H)\}$ and Prisner shows too how to have a bound of n. He use a complicates technics like homology to prove this result. We have found an easier proof of this result whit the technics in present paper, this proof is in [2, 3].

We can resume the main result in this paper as it follows: If $x \in V(G)$ is dominated then G and $G - \{x\}$ have the same $k-$ behavior.

The $k-$**behavior** is studied for many reasons, one of these, is the result of Hazan and Neumann-Lara [5]. They establish that every endomorphism in a comparison graph that is $k-$**null** has the property of fix point.

For notation see [4] for large bibliography see [2].

2 Dismantlings.

Definition 2.1 *Let G be a graph and H' a subgraph of G.*

1. *There is an innerly short dismantling of G to H' if every element of $V(G) - V(H')$ is dominated by some element of $V(H')$; and we write $G \overset{\#_0}{\to} H'$*
2. *If H is a graph, we say that there is a short dismantling of G to H when there exist H', subgraph of G, which is isomorphic to H and such that $G \overset{\#_0}{\to} H'$. In this case we write $G \overset{\#}{\to} H$, in particular $G \overset{\#}{\to} H'$.*

We could observe that $c(G) \overset{\#_0}{\to} k(G)$, where $c(G)$ is the intersecction graph of family of complete subgraphs of G.

Let us observe that G is a cone if and only if $G \overset{\#}{\to} K_1$.

3 Dismantlings and the Clique Operator

Lemma 3.1 *Assume $G \overset{\#_0}{\to} H$ and A, B are cliques of G (not necessarily different) with $A \cap B \neq \emptyset$. Then there is $y \in V(H)$ such that $y \in A \cap B$.*

Theorem 3.2 *Let G and H be graphs such that $G \overset{\#}{\to} H$. Then $k(G) \overset{\#}{\to} k(H)$.*

Let $H' \cong H$ such that $G \overset{\#_0}{\to} H'$. Let us define $g : k(H') \to k(G)$, let $B \in V(k(H'))$. Since B is a complete subgraph of G, we may choose a clique $g(B)$ of G such that $B \subseteq g(B)$. Is easy to see [5, 6] that g defines an isomorphism of $k(H')$ in its image.

Now we will prove that $k(G) \overset{\#_0}{\to} g(k(H'))$.

Let $A \in V(k(G)) - V(g(k(H')))$ and B be a clique of H' that contains $A \cap V(H')$. We will prove that $A \lesssim g(B)$:

Let $C \in N[A]$. Then $A \cap C \neq \emptyset$ and by Lemma 3.1 there is $y \in A \cap C \cap V(H')$. Thus $y \in B$, and consequently $y \in g(B)$. Therefore $g(B) \cap C \neq \emptyset$ and $C \in N[g(B)]$.

4 Main Result

Lemma 4.1 *Let G, H and H' be graphs. If $G \overset{\#}{\to} H$ and $G \overset{\#_0}{\to} H' \cong H$ we will define $G_{\#H}$ as the graph of intersection of $A \cap V(H')$ where A is a clique of G. The next staments are hold:*

1. $c(H') \overset{\#_0}{\to} G_{\#H} \overset{\#_0}{\to} k(H')$.
2. There is a retraction [5, 6] $f : k(G) \to G_{\#H}$ with inverse $g : G_{\#H} \to k(G)$ with the next properties:
   a . $N[g(f(A))] = N[A]$ for every $A \in k(G)$.
   b . $k(G) \overset{\#_0}{\to} g(G_{\#H})$.
3. $k^2(G) \cong k(G_{\#H})$.
4. If H is a periodic graph of period n, then $k^{n-1}(c(H)) \overset{\#}{\to} k^n(G) \overset{\#}{\to} H$.

1. It is clear from $k(H') \subseteq_* G_{\#H} \subseteq_* c(H')$ and $c(H') \overset{\#_0}{\to} k(H')$.
2. Let us define $f(A) = A \cap V(H')$. Let B be an element of $V(G_{\#H})$ and $g(B)$ a clique of G that contains B and such that $f(g(B)) = B$. By Lemma 3.1, f preserves edges and clearly g is an isomorphism over its image. Therefore f is retraction.

   Let $A, B \in V(k(G))$ such that $f(A) = f(B)$. By Lemma 3.1, if $C \in V(k(G))$ is a neighbor of A, then it is a neighbor of B too, and since $f(g(f(A))) = f(A)$, we have $N[g(f(A))] = N[A]$. In particular $N[A] \subseteq N[g(f(A))]$, and hence $k(G) \overset{\#_0}{\to} g(G_{\#H})$.
3. We have $G_{\#H} \cong g(G_{\#H})$, where g is as g in 2. The statment is followed of 2.
4. It is followed of 1, 2 and 3.

Lemma 4.2 *Let H be a periodic graph of period n then $c(H)$ is $k-$stationary and there is m such that $k^m(c(H)) \cong k^{m+n}(c(H))$.*

We have that $c(H) \xrightarrow{\#} k(H)$. In Lemma 4.1.4 we make $G = k^{n-1}(c(H))$ and we obtain

$$k^{n-1}(c(H)) \xrightarrow{\#} k^{2n-1}(c(H)) \xrightarrow{\#} H$$

hence

$$k^{n-1}(c(H)) \xrightarrow{\#} k^{2n-1}(c(H)) \xrightarrow{\#} k^{3n-1}(c(H)) \xrightarrow{\#} ... \xrightarrow{\#} k^{mn-1}(c(H)) \xrightarrow{\#} H$$

for every natural m, since $k^{n-1}(c(H))$ is of finite order there are two distincts naturals i_0 and i_1 such that:

$$k^{i_0 n-1}(c(H)) = k^{i_1 n-1}(c(H))$$

with $i_0 < i_1$ then

$$k^{i_0 n-1}(c(H)) \xrightarrow{\#} k^{(i_0+1)n-1}(c(H)) \xrightarrow{\#} ... \xrightarrow{\#} k^{i_1 n-1}(c(H))$$

hence

$$k^{i_0 n-1}(c(H)) = k^{(i_0+1)n-1}(c(H))$$

Let l be the period of $c(H)$. Then l divide n and there is m such that $k^m(c(H)) \cong k^{m+rl}(c(H))$.

Now we prove main result of this paper:

Theorem 4.3 *$G \xrightarrow{\#} H$ then G and H have the same $k-$behavior.*

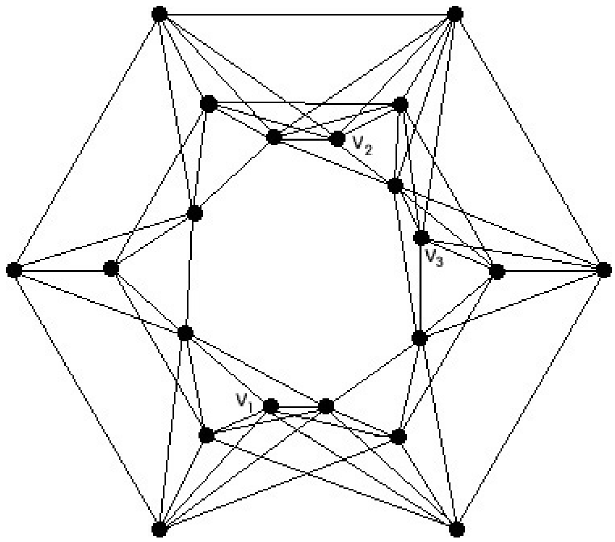
1) It is clear from the theorem 3.2.

2a) If G is stationary, then there are n, m such that
$k^n(G) \cong k^{n+m}(G) \xrightarrow{\#} k^{n+im}(H)$, for every $i \in \mathbb{N}$, but since $k^m(G)$ is dismantliable to a finite number of graphs, there aredistincts i_0 and i_1 such that $k^{m+i_0 n}(H) = k^{m+i_1 n}(H)$. Therefore H is a stationary.

2b) That H is $k-$stationary implies that G is k-stationary follows from Lemma 4.1.4, Lemma 4.2 and 2a).

3) This is a consequence of the **1)** and **2)**.

Remark 4.4 *The graph in figure 4 has a period 3 and has three dominated points v_1, v_2, v_3. If we delete v_1 we obtain a 6-periodic graph, if we delete v_3 we obtain a 1-periodic graph. Let G, H be graphs such that $G \xrightarrow{\#} H$ and there are n, and m such that $k^{m+n}(G) \cong k^m(G)$. From Theorem 4.3 there are m' and n' such that $k^{m'+n'}(H) \cong k^{m'}(H)$. The determination of the relation among m, n, m' and n' is an open problem.*



References

[1] F. Escalante. *Über Iterierte Clique-Graphen.* Abh. Math. Sem. Univ. Hamburg. 39 (1973) 58-68.

[2] M.E. Frías-Armenta. Tesis Doctoral: *Gráficas Iteradas de Clanes.* Facultad de Ciencias, Universidad Nacional Autonoma de México (2000) Dir. V. Neumann-Lara Imunam.

[3] M.E. Frías-Armenta, V. Neumann-Lara. Cliques, Prekernels, Operators, Covers and pointer graphs. In preparation.

[4] F. Harary. Graph Theory. Addison-Wesley, Reading, MA (1969).

[5] S. Hazan, V. Neumann-Lara. *Fixed Points of Posets and Clique Graph.* Kliwer Academic Publishers 13 (1996) 219-225.

[6] V. Neumann-Lara. *On Clique-divergent Graphs*, Problèmes Combinatoires et Théorie des Graphes (Colloques internationaux C.N.R.S, 260). Paris (1978), 313-315.

[7] V. Neumann-Lara. *Clique Divergence in Graphs*, Algebraic Methods in Graph Theory (Coll. Math.Soc.Janos Bolyai, 25). Szeged (1981), 563-569.

[8] E. Prisner. *Convergence of Iterated Clique Graphs.* Discrete Mathematics 103 (1992) 199-207.